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Statistical Laboratory

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SEVENTH QUARTERLY PROGRESS REPORT

PROGRESS REPORT 523.20

Field Assessment Problems

COVERING THE PERIOD

1 March - 30 May 1961

Prepared by

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ABSTRACT CARD

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Statistical Laboratory, Research Division College of Engineering, New York University, New York 53, N.Y. FIELD ASSESSMENT PROBLEMS L. Herbach, B. Davidson and I. Pomper

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1. Coverage Problems
2. Micrometeorology
3. Munition Expenditure
4. Atmospheric Diffusion

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IV. Contract DA18-108-Cml-6392/ DA18-108-405-Cml-256

1. It is shown that the Calder-Salzer technique for computing munition expenditures probably implies that the moments of the dosage distribution do not exist. A simple modification of the Salzer inversion technique is being tried which does not require "scaling". 2. The diffusion model treated heretofore is simplified by introducing dimensionless parameters. This new procedure enables one to obtain a family of depositions as the result of a single integration. 3. An outline of a new stochastic model of turbulent diffusion is given.

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1. MUNITION EXPENDITURE

In the previous progress report we indicated that the Calder Salzer technique for inverting a Laplace Transform imposes unknown conditions on the moments of the multi-munition dosage distribution.

Of course, as Calder points out a direct use of the Salzer scheme would imply that all moments are infinite. Indeed it would imply that the total area under the "distribution" is infinite. That is if we assume that the Laplace Transform,

$$(1) \qquad \mathcal{L}(t) = \int_0^\infty e^{-tD} f(D)\,dD = \sum_{i=1}^{k} \frac{b_i}{t^i} ,$$

then the dosage distribution is

$$(2) \qquad f(D) = \sum_{r=1}^{k} \frac{b_r D^{r-1}}{(r-1)!} .$$

Clearly if $f(D)$ is given by (2) then

$$\int_0^\infty f(D)\,dD = \infty$$

and $f(D)$ cannot be a probability density. Actually Calder uses the Laplace Transform of the cumulative distribution function, so that one must replace $f(D)$ in (1) and (2) by $F(D)$. As he points out

$$(3) \qquad F(D) = \sum_{r=1}^{k} \frac{b_r D^{r-1}}{(r-1)!}$$

implies that $F(\infty) = \infty$ and not 1, as we must have for any cumulative distribution function. Differentiating (3) we have

$$f(D) = \sum_{r=2}^{k} \frac{b_r D^{r-2}}{(r-2)!} \quad , \tag{4}$$

from which we draw the same conclusion. Another difficulty with (2) is that the polynomial goes through (0,0). Thus it implies zero probability of obtaining dosages in the neighborhood of zero.

Calder handles both objections as follows. To get around the non-zero probability of zero coverage he truncates the dosage distribution at the lower end. To get around the unboundedness of the polynomial at $D = \infty$ he introduces a scaling factor (so that the modified $F(D)$ is bounded by 1). Thus, in effect he uses a different polynomial fit for each D-value. It is difficult to estimate the effect that this modification has on the original physical problem. However, it is reasonable to assume that a family of polynomials can be made to fit a function better than a single polynomial.

In addition, although the scaling technique enables one to fit an $F(D)$ with the property that $F(D) \to 1$ as $D \to \infty$, one would probably need a stronger scaling factor to guarantee that all moments exist. One still has

$$\lim_{A \to \infty} \int_0^A D^i f(D) dD = \infty \ ,$$ †

so that this method really implies that the moments of the dosage distribution do not exist.

It should also be pointed out that the theoretical justification for fitting $f(D)$ by a polynomial is related to the Weierstraβ Approximation Theorem which states that any continuous function can be closely approximated by a polynomial in D for real D satisfying $0 < a \leq D \leq b$. Notice that this theorem holds for functions over <u>finite</u> intervals, but that we are dealing with Laplace transforms which are integrals defined over an infinite range. We can then show that by using the unscaled Laplace Transform method it is not possible to "fit" the bounded functions by a polynomial. To rigorize the scaling technique we need, a corrollary to the Weierstraβ Theorem, which shows that a sequence of polynomials will do the trick. The Calder technique employs a sequence of polynomials, each of which fits at one of the sequences of D-values.

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† In the last progress report (p.2) we wrote the Laplace Transform in terms of moments of the distribution. In analogy with the moment generating function we should have used

$$\mathcal{L}(t) = \sum_{i=0}^{\infty} (-1)^i \frac{\mu_i t^i}{i!} \ .$$

The $(-1)^i$ was inadvertently omitted.

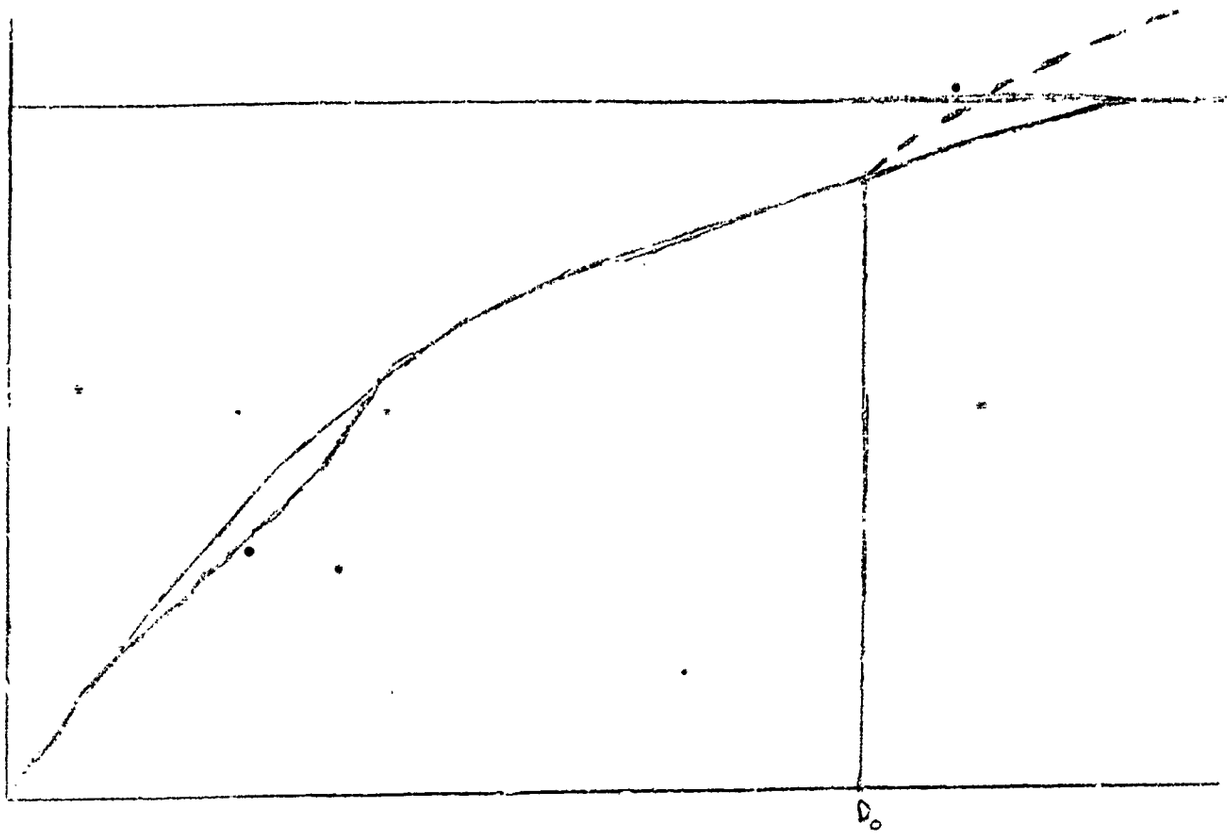

Figure 1. Polynomial approximation to F(D)

We are now in the process of trying a less sophisticated technique than scaling to allow for the fact that a polynomial is infinite at $D = \infty$. It consists in fitting the polynomial between $D = 0$ and $D = D_0$ and then arbitrarily approximating F(D) by

$$F(D) = 1 \quad , \quad D > D_0 \ .$$

If D_0 is large enough so that we are really not interested in the probability of getting dosages above D_0 and if our approximation gives $F(D) \leq 1 \quad D \leq D_0$, this procedure may be satisfactory.

2. ATMOSPHERIC DIFFUSION

In an attempt to simplify the diffusion model treated in these reports a version which depends on dimensionless parameters is proposed. As a by-product of this new procedure we are enabled to obtain a family of depositions by multiplying by simple expressions the result of integrations, similar to (10) and (11) of the previous progress report.

Our deposition of agent, K as a function of distance downwind, x was given by

$$K(x) = A \int_{\Omega_1}^{\Omega_2} h(\Omega)t(x,\Omega)\,d\Omega \tag{5}$$

where

$$h(\Omega;\mu,t) = \frac{1}{(2\pi)^{\frac{1}{2}}\, t} \exp\left[-\frac{(\ell n\Omega-\mu)^2}{2t^2}\right] \tag{6}$$

$$t(x,\Omega) = \frac{1}{\Gamma(1+p)} \left(\frac{f}{x}\right)^{1+p} e^{-f/x} \tag{7}$$

$$p = c\Omega \tag{8}$$

We now make two changes. In place of x we consider the dimensionless parameter x/f . For any value of f it is then a simple matter to treat K as a function of x. Secondly the integration is performed with p, rather than Ω as the variable of integration. Thus

(9) $$\frac{K}{Ac} = \int_0^\infty h(p;\varphi,t)\, v(r,p)dp = M(r;\varphi,t) \text{ , say}$$

where

(10) $$r = x/f$$

(11)
$$h(p;\varphi,t) = \frac{1}{\sqrt{2\pi}\, t} \exp\left[\frac{\{\ell np + (\ell nc - \mu)\}^2}{2t^2}\right]$$
$$= \frac{1}{\sqrt{2\pi}\, t} \exp\left[-\frac{\{\ell np - (\varphi)\}^2}{2t^2}\right]$$

(12) $$\varphi = \mu - \ell nc$$

(13) $$v(r,p) = \left[\Gamma(1+p)\, r^{1+p}\, e^{r^{-1}}\right]^{-1} .$$

The meteorological parameters are incorporated in A, c and $f = Q/(Ac)$. Thus, each curve M can be easily transformed by (10), (11) and (12) to a set of curves, K, for specific meteorological conditions and particle size distributions. The conditions treated in previous reports are given by $A = .1674 \frac{gm}{m^2} / \frac{cm}{sec}$ and the values in Table 1. It should be noted that in all these cases φ, the mean of the "normal" distribution in (11) was negative. Since φ, rather than μ seems to be the pertinent "mean" to consider in evaluating K it was decided to compute some values for $\varphi=0$ and $\varphi=1$, where we expect diffusion

to play a larger role. For this reason M_φ of (9) is being evaluated on the IBM 650 for the parameters of Table 2 by Mr. Daniel H. Salkoff, who performed the previous calculations.

f	μ	c^{-1}	ℓnc	t	φ
514	4.06	.08603	1.7595	.53	+ 2.30
514	3.26	.08603	1.7595	.57	+ 1.50
1028	4.06	.17206	2.453	.53	+ 1.61
1028	3.26	.17206	2.453	.57	+ .81

Table 1. Parameters in previous deposition calculations.

φ	t	p_1	p_2
0	.3	.40	2.5
0	.55	.19	5.2
0	1.0	.05	20.0
-1	.55	~~.522~~ .071	~~14.154~~ 1.92

Table 2. Values of parameters being used in computing M as given in Equation (14), where $p_i \approx e^{\varphi \pm 3r}$.

This new procedure cuts out the necessity for evaluating L, the integral with p and f (or r) replaced by $2p$, $2f$ (or $\frac{r}{2}$). If one actually evaluated (9) exactly, the change in these parameters would merely change the integrating grid and give different values of distance downwind. However, we, of course, replace (9) by

$$(14) \qquad M = \frac{K}{Ac} \approx \int_{p_1}^{p_2} h(p,\varphi,t)\, v(r,p)dp$$

where p_1 and p_2 are chosen so that the contribution to M of values where p_2 and below p, are negligible. Replacing p by $2p$ would also change the limits of integration, thereby omitting important values of p and replacing them by values which make virtually no contribution to M.

An outline of a new stochastic model of turbulent diffusion is given in an appendix to this report. Of course, the details are yet to be programmed for computation on a digital computer, and these results would have to be compared with field test data to see how well the model fits.

Leon H Herbach

Leon H. Herbach
Project Director

APPENDIX

Memo 523.13 M.A.W./L.H./I.P. April 1961
May 1961

A stochastic model of turbulent diffusion

The purpose of this memorandum is to formulate a new model of turbulent diffusion by a simulation of the physical process. The model will describe the process during a fixed instant of time. In more picturesque language, it may be described as being a "snapshot" of the physical process. It is hoped in later reports to describe a continuous model for the same physical phenomena.

It is assumed that an instantaneous point source located at $x = x_0 = 0$, $y = y_0 = 0$ originates as a "mass" Q_0 which may be described mathematically by the delta function, or unit impulse function, $Q_0\,\delta(x,y)$ where $\delta(x,y)$ is defined by

$$\delta(x,y) = 0 \quad \text{if} \quad x \neq 0 \quad \text{or} \quad y \neq 0$$

$$\delta(x,y) = \infty \quad \text{if} \quad x = 0 \quad \text{and} \quad y = 0 \tag{1}$$

$$Q_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(x,y) = Q_0$$

Our next assumption is that the mass will diffuse according to a "random walk" process whose mechanism is described below. We shall be particularly interested in studying the correlation functions of the stochastic process involved. These correlation functions will be used to describe the rate of diffusion, and the general features of the process mechanism. The correlation

function as generated will consider the relations between fixed discrete points.

It is assumed that the "mass" undergoes a splitting which gives rise to two "daughter" masses. Each of these daughters has a total "mass" $Q_0/2$ and is centered at the points

$$(2) \qquad x_1^{(i)} = x_0 + \epsilon^{(i)} \; ; \; y_1^{(i)} = x_0 + \delta^{(i)} \qquad i=1,2$$

where $\epsilon^{(i)}$ and $\delta^{(i)}$ are normally and independently distributed with means zero and variances σ_1^2 .

The "splitting" process continues in this fashion except that the variances of the new ϵ's and δ's are σ_i^2 , $i = 2,3,4,5$, with $\sigma_5^2 < \sigma_4^2 < \sigma_3^2 < \sigma_2^2 < \sigma_1^2$, until we arrive at the points $x_5^{(i)}$, $y_5^{(i)}$ where $(i = 1,2,\ldots,32.)$ The process is illustrated in Figure 1. That is, the process continues until we have generated 2^5 or 32 points.

The effect of wind can be taken into account by this model. If a constant wind is blowing from some direction, its effect on the distribution of the sample points can be allowed for by assuming that the mean of the first points ϵ_1 , δ_1 $(i=1,2)$ are non-zero. For example if the wind is from the SW we may take $E(\epsilon_1) = 1$, $E(\delta_1) = 1$.

At each of the 32 points which have been generated we now conceive of the mass, $Q_0/2^5$ as being distributed according

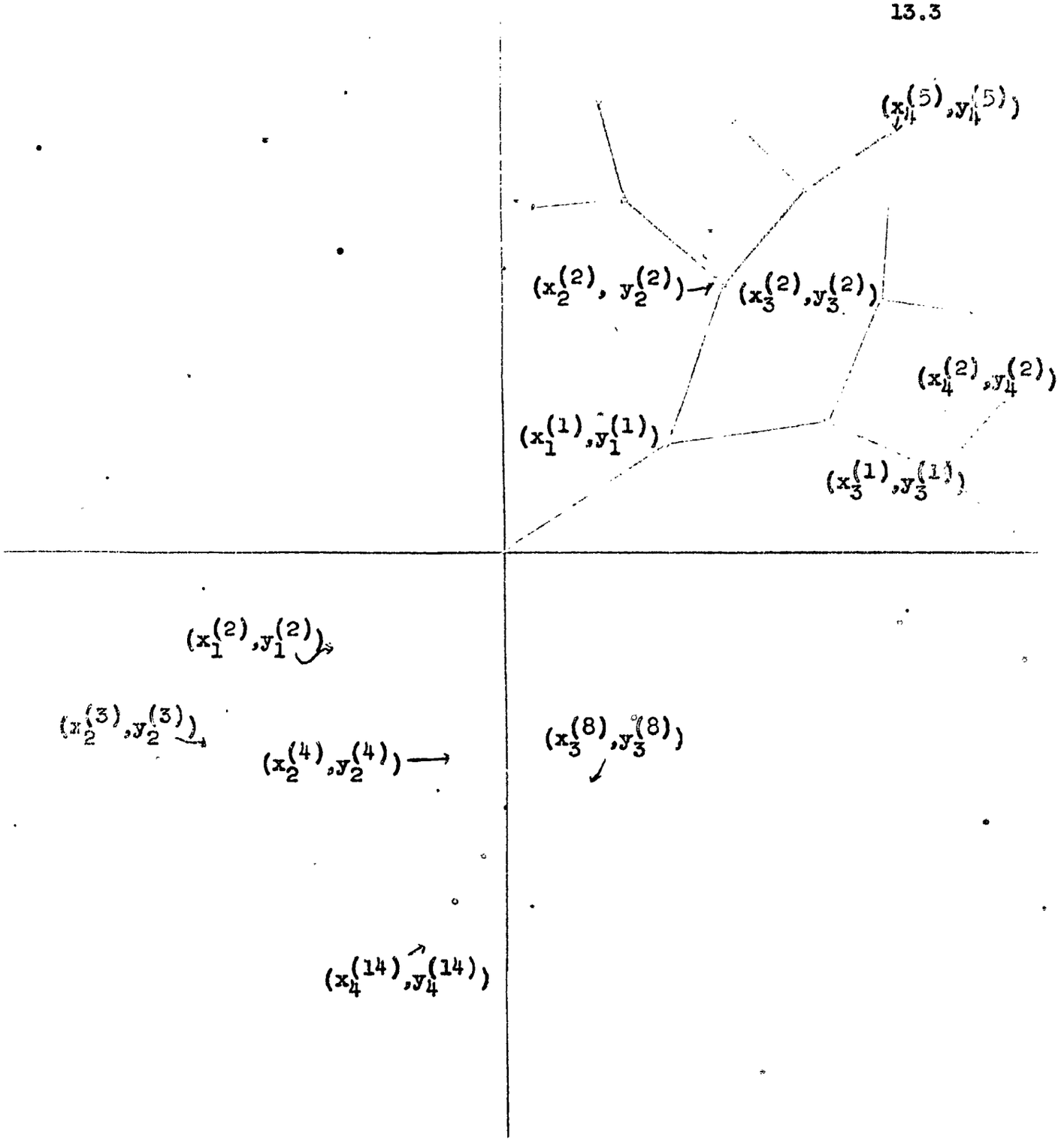

Figure 1. The Splitting Process Generating the Random

to a squared exponential. This is the limiting result if we continue the splitting ad infinitum with small variances. Thus, the density at any point (u,v) in space is

$$(3) \qquad \sum_{i=1}^{32} \frac{Q/2^n}{(2\pi)\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\left[(u - x_5^{(i)})^2 + (v - y_5^{(i)})^2\right]\right)$$

where the points are generated in the following manner:

$$\begin{aligned}
&(x_0,y_0) \to (x_1^{(1)}, y_1^{(1)}) \ , \ (x_1^{(2)}, y_1^{(2)}) \\
&(x_1^{(1)}, y_1^{(1)}) \to (x_2^{(1)}, y_2^{(1)}) \ , \ (x_2^{(2)}, y_2^{(2)}) \\
&(x_1^{(2)}, y_1^{(2)}) \to (x_2^{(3)}, y_2^{(3)}) \ , \ (x_2^{(4)}, y_2^{(4)}) \\
(4) \qquad &(x_2^{(1)}, y_2^{(1)}) \to (x_3^{(1)}, y_3^{(1)}) \ , \ (x_3^{(2)}, y_3^{(2)}) \\
&(x_2^{(2)}, y_2^{(2)}) \to (x_3^{(3)}, y_3^{(3)}) \ , \ (x_3^{(4)}, y_3^{(4)}) \\
&(x_2^{(3)}, y_2^{(3)}) \to (x_3^{(5)}, y_3^{(5)}) \ , \ (x_3^{(6)}, y_3^{(6)}) \\
&(x_2^{(4)}, y_2^{(4)}) \to (x_3^{(7)}, y_3^{(7)}) \ , \ (x_3^{(8)}, y_3^{(8)}) \ , \text{ etc.}
\end{aligned}$$

The reason for using a squared exponential as the limiting distribution for the "splitting" of steps 6 to infinity, even though the resulting distribution is singular, is as follows: The average amount in any region after an infinite number of steps is given by the circular normal integral. We are in effect replacing the results after step five, by their averages. It is clear, from the way, the variances of subsequent steps decrease that the variability in these latter steps is negligible compared with the earlier steps. Actual calculations will indicate whether

a good approximation will be obtained if this "averaging" is done after step 5, 6, or some other number.

It should also be noted that we have still not taken time into account, since all stages in the splitting process are assumed to take place instantaneously. Time is considered by replacing σ^2 in (3) by $\sigma^2 f(t)$, where $f(t)$ is an increasing function of time. The first function to be considered will be $f(t) = t$.

Now we may consider the correlation structure. It may be noted that the σ's which must be used here will have to be obtained experimentally. The correlation structure for the first two steps is given as

$$R_0 = \begin{bmatrix} \sigma_0^2 & 0 \\ 0 & \sigma_0^2 \end{bmatrix} = \sigma_0^2 I_2$$

(9)

$$R_1 = \sigma_1^2 I$$

In considering the diagram shown in Figure 2 it is noted that if the correlation structure is given in the following manner of describing the splitting process we have

(10) $$x = x_0 = 0 \quad , \quad y = y_0 = 0 \text{ with } \sigma_0$$

as the splitting continues we obtain at the first step the following σ's, where x_0 = first mass and x_1, the second mass, considering all possible combinations

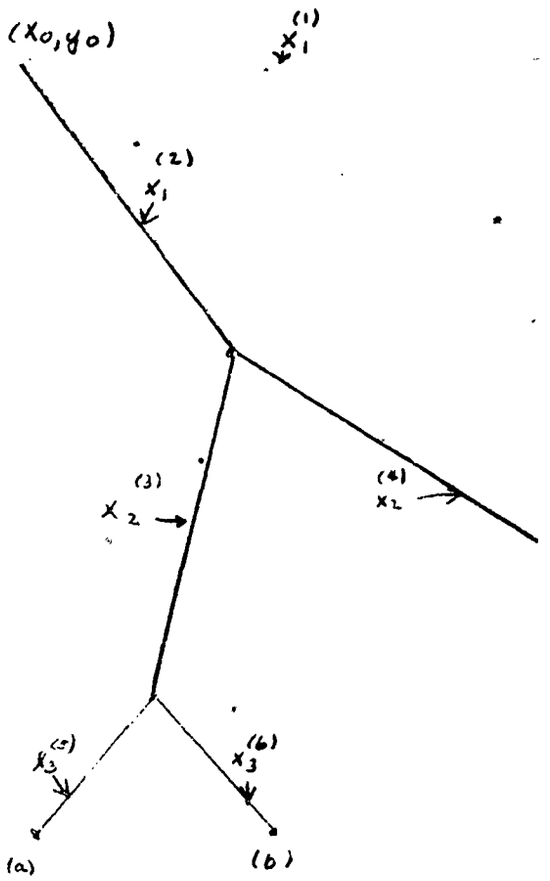

Figure 2. Development of Covariance Structure

(11)

x_{00}	x_{01}	x_{10}	x_{11}
$\sigma_0^2 + \sigma_1^2$	σ_0^2	0	0
	$\sigma_0^2 + \sigma_1^2$	0	0
		$\sigma_0^2 + \sigma_1^2$	σ_0^2
			$\sigma_0^2 + \sigma_1^2$

and at the next step we obtain the following:

(12)

x_{000}	x_{001}	x_{010}	x_{011}	x_{100}
$\sigma_0^2 + \sigma_1^2 + \sigma_2^2$	$\sigma_0^2 + \sigma_1^2$	σ_0^2	σ_0^2	0
	$\sigma_0^2 + \sigma_1^2 + \sigma_2^2$	σ_0^2	σ_0^2	0
		$\sigma_0^2 + \sigma_1^2 + \sigma_2^2$	$\sigma_0^2 + \sigma_1^2$	0
			$\sigma_0^2 + \sigma_1^2 + \sigma_2^2$	0

etc.

Now consider the correlations of these realizations:

(13) At $R_0 = \begin{bmatrix} \sigma_0^2 & 0 \\ 0 & \sigma_0^2 \end{bmatrix} = \sigma_0^2 I_2$ (identity matrix)

$$R_1 = \sigma_1^2 I_2 \cdot I_2 + \sigma_0^2 \begin{bmatrix} E_2 & 0 \\ 0 & E_2 \end{bmatrix}$$

where

$$E_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

or

$$R_1 = E_2 \cdot R_0 + \sigma_1^2 I_2 \cdot I_2$$

(14)

$$R_2 = \left[\sigma_0^2 E_2 \cdot E_2 + \sigma_1^2 E_2 \cdot I_2 \right] + \sigma_2^2 I_2 \cdot I_2$$

$$R_2 = E_2 \cdot R_1 + \sigma_2^2 I_2 \cdot I_2 \cdot I_2 \quad .$$

It will be noted that the covariance matrix at any given realization of the process gives the history of the "mass" up to this point, and essentially describes the fact that a high correlation will be found for points which lie in a close relation physically to each other. While they have a low correlation with the "mass" points which lie at some distance from each other. This is noted in Figure 2 and seems to agree with the physical theory.

To end the matter of notation we will designate the points at any given realization in their natural order. At the fifth step we will number the realizations as $x_1, x_2, \ldots, x_{32}$.

Of course, the details of this model still have to be worked out. Among the problems to be investigated are: (i) How can one obtain the mean variance and the distribution of dosage at a point or set of points? (ii) What is the limiting distribution of dosage at a point? If we drop squared exponentials uniformly over a large target the resulting distribution is c.d. We will probably be involved with a distribution which is the sum of c.d's.